Domination in Diameter-Two Graphs

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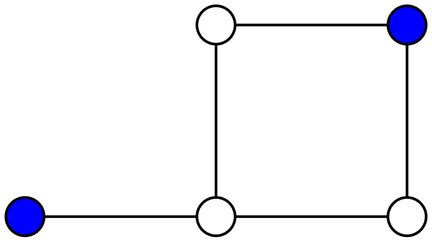
Domination Problems

- A **dominating set** (DS) is a subset X of a graph such that each vertex not in X is connected to a vertex of X
- The minimum size of a dominating set is called the **domination number** of G.

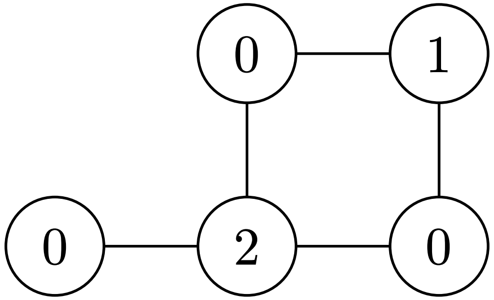


Example of DS in blue (also an IDS and an EDS).

- **IDS** (for Independent DS): the subset must be independent (no edge between two vertices of X)
- **EDS** (for Efficient DS): the distance between each pair of vertices in the subset must be at least 3. *In this case, each vertex is dominated exactly once.*

Domination Problems

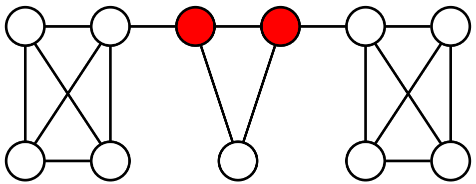
- **RD** (for **Roman Domination**): find a function $f : V(G) \to \{0, 1, 2\}$ such that each vertex labeled 0 is adjacent to a vertex labeled 2 minimizing $\sum_{v \in V(G)} f(v)$.
- **IRD** (for Independent RD): a RD function such that $f^{-1}(\{2\})$ is an independent set.
- **PRD** (for Perfect RD): an RD function such that the vertices of $f^{-1}(\{2\})$ are at distance at least 3 from each other.



Example of an RD function of weight 3.

- A set X of vertices is said to be a **CVD set** if $G[V \setminus X]$ is a disjoint union of cliques.

  It is NP-complete to compute the minimum size of a CVD set.

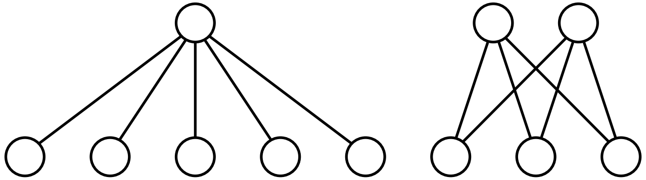


Question

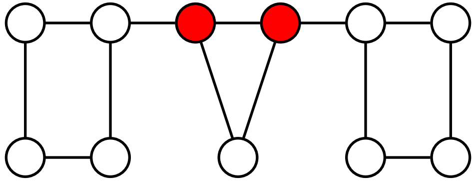
EDS parameterized by CVD is FPT. What if we relax to **less dense graphs** than cliques?

2-Club Cluster Vertex Deletion (2CCVD)

- A 2-club is a graph of diameter at most 2.



- A 2CCVD is a subset of vertices of a graph such that the removal of them gives an union of 2-clubs.



Our Results

Problems	Complexity on 2-clubs	Param. by 2ccvd
DS, CDS	NP-hard	para-NP-hard
IDS, RD, IRD	W[1]-hard	para-NP-hard
EDS	Linear-time	FPT
PRD	NP-hard	para-NP-hard

EDS parameterized by 2CCVD

Theorem

EDS parameterized by 2CCVD is FPT

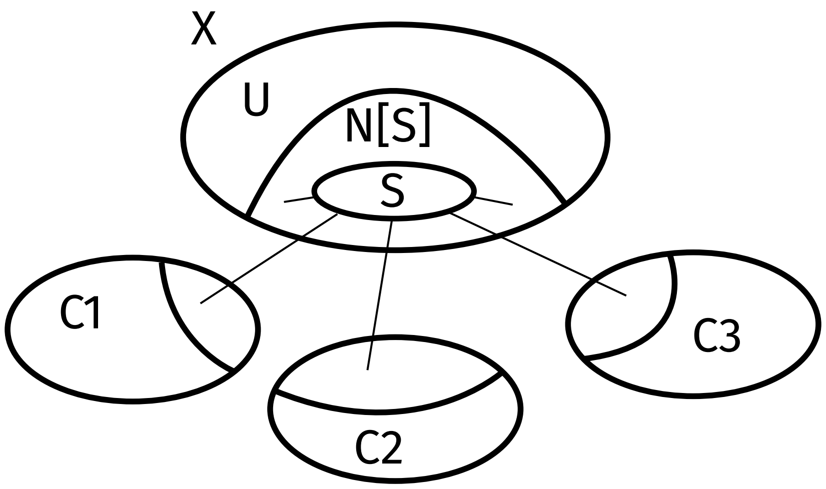
Proof by dynamic programming

EDS parameterized by 2CCVD

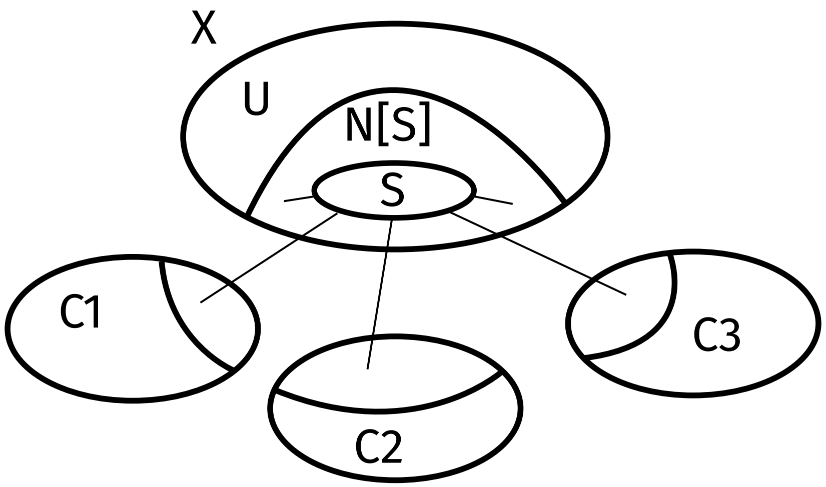
Sketch of the proof

Suppose that we have a 2CCVD subset X of size k. Let $S \subseteq X$. Can we extend S to an EDS of G?

- $N[S]$ is the closed neighborhood of S
- U is complementary of $N[S]$ in X
- the C_i's are the vertices of the 2-clubs not adjacent to S



EDS parameterized by 2CCVD



For each $W \subseteq U$, we define $T[W, j]$ as a boolean which is true if there exists vertices $v_1, \ldots, v_j$ in $C_1 \times \cdots \times C_j$ such that

- they are at distance at least 3 from each other
- they dominate exactly W and the C_i's

S is extendable $\Leftrightarrow T[U, p]$ is true

EDS parameterized by 2CCVD

Theorem

EDS can be computed in $3^k O(n^3)$ where k is the size of a 2CCVD set.

Proof

Considering all the subsets of X is in 2^k.

Try to extend such a subset is in $\left(\frac{3}{2}\right)^k n^3$.

Theorem

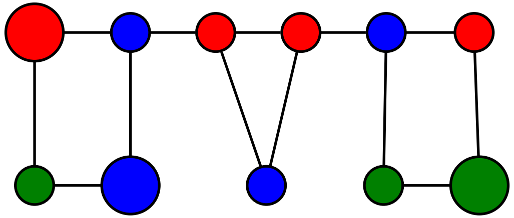
IDS is W[1]-hard on 2-clubs

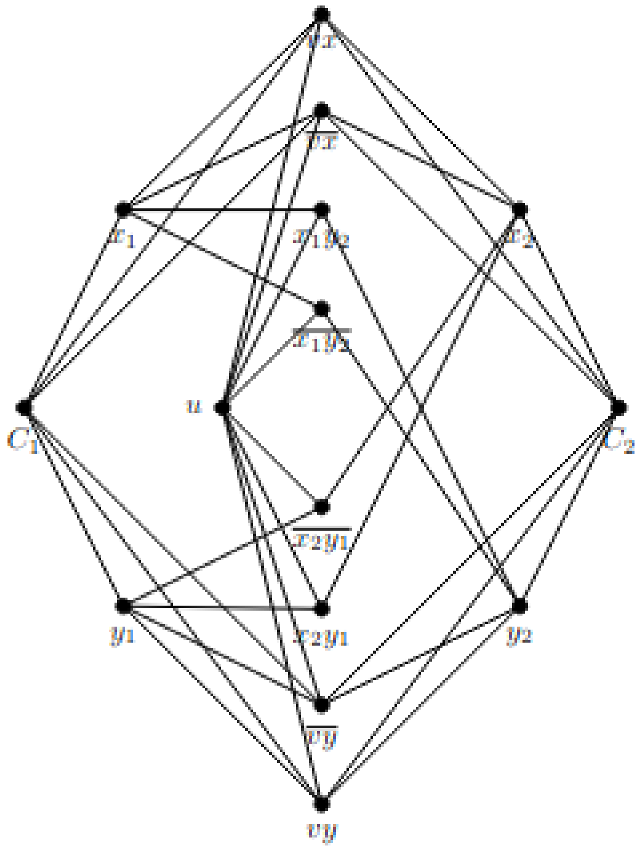
Proof by a parameterized reduction from

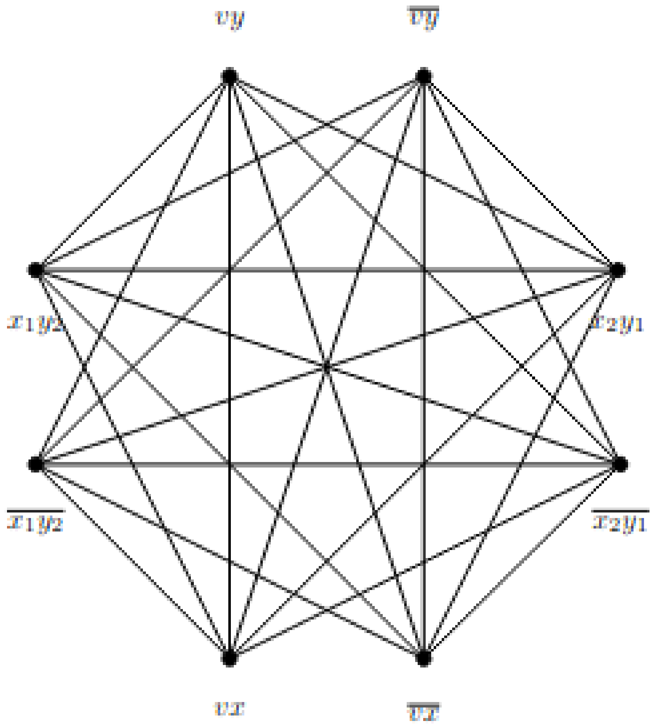
k-MULTICOLORED INDEPENDENT SET

Input: A graph G and a vertex coloring $c : V(G) \to \{1, 2, ..., k\}$.

Question: Does G have an independent set including vertices of all k colors?







Parameterized reduction.

FP sub-exponential algorithms

Starting from the fact that the minimum size of a DS in a 2-club is $O\left(\sqrt{n \log(n)}\right)$:

Theorem (Mertzios, Spirakis)

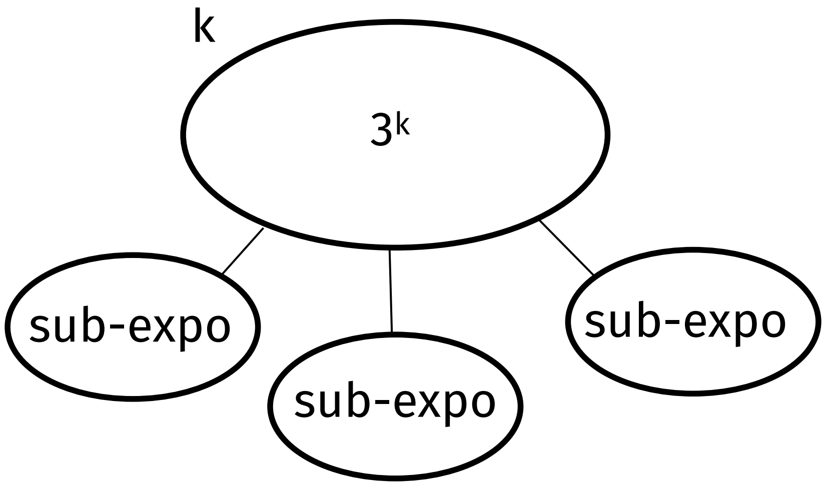
3-COLORING in 2-clubs in $O\left(3^{c\sqrt{n \log(n)}}\right)$

Theorem (Debski et al.)

3-COLORING in 2-clubs in $2^{O\left(n^{\frac{1}{3}} \log^2 n\right)}$

FP sub-exponential algorithms

Corollary: let X be a 2CCVD of size k:



Theorem

3-COLORING in $O\left(3^k 2^{cn^{\frac{1}{3}} \log^2 n}\right)$ where parameterized by $k = 2\text{ccvd}$

FP sub-exponential algorithms

Theorem

3-COLORING in $O\left(3^k 2^{cn^{\frac{1}{3}} \log^2 n}\right)$ where parameterized by $k = 2\mathrm{ccvd}$

Question

Which other problems have an FP sub-exponential algorithm parameterized by 2ccvd?

More generally

Which problems have an FP sub-exponential algorithm?

We introduce the class **FPSUB** which consists in all such problems.

Thank you for your attention!